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Special examples of diffusions in random environment

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Abstract

In this note we present some examples of diffusions in random environment whose asymptotic behavior is rather surprising. We construct a family of diffusions that are small perturbations of Brownian motion with non-vanishing expected local drift under the static measure of the environment but where the ballistic behavior is lost. As slight modifications of this collection of diffusions we also provide examples with ballistic behavior where the non-vanishing limiting velocity points to a direction opposite to the expected local drift under the static measure.

MSC: 82D30; 60K37

Keywords: Diffusions in random environment; Invariant ergodic measure; Environment viewed from the particle

1. Introduction

In this note we construct examples of diffusions in random environment which are small perturbations of Brownian motion and behave diffusively although their expected local drift does not vanish. We also provide examples where the diffusion has non-degenerate asymptotic velocity but the average local drift vanishes, as well as examples where the asymptotic velocity points in the opposite direction to the average local drift. These examples show that the naive guess that the average local drift controls whether the diffusion is ballistic (i.e. has non-degenerate velocity) or not, is wrong, even for small perturbations of Brownian motion. The crucial tool in the construction of the above examples is the so-called method of *the environment viewed from the particle*, see below (1.9) for more comments and references on this technique.

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Let us mention that there has recently been an intense activity around the investigation of diffusions as well as random walks in random environment, notably in the study of ballistic behavior, see [5,6,17–22,26]. As for diffusive behavior, some progress has also been made, see [1, 3,25]. For an overview of results and useful techniques concerning this area of research we also refer to [12,23,24,28,29].

We now describe the model. We consider dimensions $d \geq 2$ and the random environment is described by a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We write $E^{\mathbb{P}}$ for the expectation with respect to the measure $\mathbb{P}$. We assume the existence of a group $\{\tau_x : x \in \mathbb{R}^d\}$ of $\mathbb{P}$-preserving transformations which act ergodically on Ω and are jointly measurable in $x \in \mathbb{R}^d, \omega \in \Omega$. Note that in what follows each measurable function of the form $f(x, \omega), x \in \mathbb{R}^d, \omega \in \Omega$, is supposed to be generated from a measurable map $\tilde{f}$ on Ω by the action $\{\tau_x : x \in \mathbb{R}^d\}$, i.e.

$$f(x, \omega) \stackrel{\text{def}}{=} \tilde{f}(\tau_x(\omega)), \tag{1.1}$$

and hence is a stationary random field. For all such functions f and all Borel subsets $F \subseteq \mathbb{R}^d$ we define the σ-algebra

$$\mathcal{H}_F^f \stackrel{\text{def}}{=} \sigma\left(f(x, \omega) : x \in F\right). \tag{1.2}$$

The diffusions under consideration in this work all have the identity as diffusion matrix and the drift is a stationary uniformly bounded function $b(x, \omega), x \in \mathbb{R}^d, \omega \in \Omega$, such that there exists a constant $\kappa > 0$ such that for all $x, y \in \mathbb{R}^d, \omega \in \Omega$, the following Lipschitz condition is satisfied:

$$|b(x, \omega) - b(y, \omega)| \leq \kappa |x - y|, \tag{1.3}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. Furthermore, the coefficient b satisfies the finite range dependence condition with range $R > 0$ that is for Borel subsets A and B of $\mathbb{R}^d$,

$$\mathcal{H}_A^b \text{ and } \mathcal{H}_B^b \text{ are } \mathbb{P}\text{-independent whenever } d(A, B) > R, \tag{1.4}$$

where $d(A, B) \stackrel{\text{def}}{=} \inf\{|x - y| : x \in A, y \in B\}$. We write $(X_t)_{t \geq 0}$ for the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$, the space of continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$. Due to boundedness and regularity of b (see (1.3)), for any $x \in \mathbb{R}^d, \omega \in \Omega$, the martingale problem attached to

$$\mathcal{L}^{\omega} = \frac{1}{2}\Delta + b(\cdot, \omega) \cdot \nabla \tag{1.5}$$

and starting in x at time 0 is well posed, see Corollary 5.4.29 in [8] for the existence of a unique solution of the above martingale problem. The law $P_{x,\omega}$ on $C(\mathbb{R}_+, \mathbb{R}^d)$ denotes its unique solution and describes the diffusion in the environment ω and starting from x. $P_{x,\omega}$ is usually called the *quenched* law and we write $E_{x,\omega}$ for the corresponding expectation. For the study of the asymptotics of $(X_t)_{t \geq 0}$, it is convenient to introduce also the *annealed* law

$$P_x \stackrel{\text{def}}{=} \mathbb{P} \times P_{x,\omega}, \quad x \in \mathbb{R}^d. \tag{1.6}$$

We denote with E_x the corresponding expectation.

Let us now explain more precisely the purpose of this work. We are going to construct in dimension $d \geq 2$ a family of small perturbations of Brownian motion attached to a second-order elliptic operator of the form (1.5) which contains examples of diffusions with arbitrarily small drifts such that the expected local drift under the static measure does not vanish but the ballistic behavior is lost. More precisely, for all $\varepsilon > 0$ small enough we find examples with $|b(x, \omega)| \le \varepsilon$ for all $x \in \mathbb{R}^d$, $\omega \in \Omega$, such that

$$E^{\mathbb{P}}[b(0, \omega)] \neq 0 \quad \text{but} \quad P_0\text{-a.s.}, \lim_{t \to \infty} \frac{X_t}{t} = 0, \tag{1.7}$$

see Theorem 3.1. As a slight modification of this class of diffusions, we will also provide examples with arbitrarily small drifts that exhibit ballistic behavior when the expected local drift vanishes, that is

$$E^{\mathbb{P}}[b(0, \omega)] = 0 \quad \text{but} \quad P_0\text{-a.s.}, \lim_{t \to \infty} \frac{X_t}{t} = v \neq 0, \tag{1.8}$$

see Theorem 3.4, with a deterministic velocity $v \neq 0$, or even has the opposite direction to the non-vanishing limiting velocity which means that there exists a positive constant $\gamma > 0$ such that

$$P_0\text{-a.s.}, \quad \lim_{t \to \infty} \frac{X_t}{t} = -\gamma E^{\mathbb{P}}[b(0, \omega)] \neq 0, \tag{1.9}$$

see Theorem 3.2. For the construction of examples with similar behavior in the case of random walks in random environment, the authors of [1] exploit the presence of the so-called cut times to derive a law of large numbers, see also del Tenno [3] for a similar technique in the continuous space-time setting. In this work the main strategy to provide examples of the nature described in (1.7)–(1.9) is the method of *the environment viewed from the particle*. For further successful applications of this method see for instance [9–11,13–16]. This technique relies on the existence of an invariant ergodic measure for the process of the environment viewed from the particle, as defined in (A.3), which is absolutely continuous with respect to the static measure of the environment and produces a law of large numbers with explicit formula for the limiting velocity. Unfortunately, for general diffusions in random environment, the problem of finding such a measure seems to be intractable. For our purpose however, we can restrict ourselves to drifts with a special structure for which the invariant measure is known, see (A.1) and Theorem A.1 in Appendix A.

The construction of the above mentioned examples using the technique of *the environment viewed from the particle* shows us that the limiting velocity of the particle is governed by the environment viewed from the particle and not by the static environment which may be different.

Let us explain how this work is organized. In Section 2 we construct a family of small perturbations of Brownian motion which contains examples of each type of behavior described above in (1.7)–(1.9). In Section 3 we then provide these examples. Section 4 contains auxiliary results and is dedicated to the proof of Lemma 2.3 stated in Section 2 and a concrete example of possible functions which are involved in the construction of the family of drifts in Section 2. Finally, in Appendix A, we will present more details on the existence of an invariant ergodic measure for the process of the environment viewed from the particle for a special class of diffusions.

2. Main construction of the drifts

In this section we are going to construct a family of small drifts such that the class of diffusions generated via the operators given in (1.5) for this family of drifts contains examples of diffusions with arbitrarily small drift which behave as described in the introduction (see (1.7)–(1.9)).

Let us first introduce some notation. For a Borel subset $A \subseteq \mathbb{R}^d$ and a real number $r > 0$ we define the open r-neighborhood of A as

$$A^r \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^d | d(x, A) < r \right\}, \tag{2.1}$$

with the distance function as defined below (1.4). For the jth, $j \geq 0$, partial derivative with respect to $x \in \mathbb{R}^d$ of a measurable function $f(x, \omega)$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, we will write

$$\partial_i^j f(x, \omega) \stackrel{\text{def}}{=} \frac{\partial^j}{\partial x_i^j} f(x, \omega), \quad i = 1, \ldots, d. \tag{2.2}$$

For simplicity we will only write ∂_i for ∂_i^1, $i = 1, \ldots, d$. Furthermore, for integer $k \geq 0$, we denote with $C^k(\mathbb{R}^d)$ the space of all k-times continuously differentiable real-valued functions on $\mathbb{R}^d$, whereas $\mathrm{Lip}_m^k(\mathbb{R}^d \times \Omega)$ for $m \geq 0$ stands for the space of all measurable functions $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}$ with $f(\cdot, \omega) \in C^k(\mathbb{R}^d)$ for each $\omega \in \Omega$ and such that for all $x, y \in \mathbb{R}^d$, $\omega \in \Omega$ and $i = 1, \ldots, d$, $j = 0, 1, \ldots, k$,

$$|\partial_i^j f(x, \omega)| \leq m, \qquad |\partial_i^j f(x, \omega) - \partial_i^j f(y, \omega)| \leq m|x - y|. \tag{2.3}$$

Note that no such control on mixed derivatives is needed in what follows.

For a function

$$\varphi \in \mathrm{Lip}_1^2(\mathbb{R}^d \times \Omega) \tag{2.4}$$

which satisfies the finite range dependence condition with range $R/2$, i.e. for all Borel subsets $A, B \subseteq \mathbb{R}^d$,

$$\mathcal{H}_A^\varphi \text{ and } \mathcal{H}_B^\varphi \text{ are } \mathbb{P}\text{-independent whenever } d(A, B) > R/2, \tag{2.5}$$

where $\mathcal{H}_A^\varphi, \mathcal{H}_B^\varphi$ are defined in (1.2), we define for $0 \leq \varepsilon \leq 1$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\phi_\varepsilon(x, \omega) \stackrel{\text{def}}{=} \frac{1 + \frac{\varepsilon}{d} \varphi(x, \omega)}{1 + \frac{\varepsilon}{d} E^{\mathbb{P}}[\varphi(0, \omega)]}. \tag{2.6}$$

Let us collect some useful properties of the family ϕ_ε, $0 \leq \varepsilon \leq 1$, in the following

Lemma 2.1. *For every Borel subset $A \subseteq \mathbb{R}^d$ we have that for all $0 \leq \varepsilon \leq 1$,*

$$\mathcal{H}_A^{\phi_\varepsilon} \subseteq \mathcal{H}_A^\varphi. \tag{2.7}$$

Moreover, the following properties hold true for all $0 \leq \varepsilon \leq 1, x, y \in \mathbb{R}^d, \omega \in \Omega$ and $i = 1, \ldots, d$: (note that $d \geq 2$)

$$\phi_\varepsilon \in \mathrm{Lip}_3^2(\mathbb{R}^d \times \Omega); \tag{2.8}$$

$$\frac{1}{3} \leq \frac{d - \varepsilon}{d + \varepsilon} \leq \phi_\varepsilon(x, \omega) \leq \frac{d + \varepsilon}{d - \varepsilon} \leq 3; \tag{2.9}$$

$$|\partial_i^j \phi_\varepsilon(x, \omega)| \leq \frac{\varepsilon}{d - \varepsilon} \leq 1 \quad \text{for } j = 1, 2; \tag{2.10}$$

$$|\partial_i^j \phi_\varepsilon(x, \omega) - \partial_i^j \phi_\varepsilon(y, \omega)| \leq \frac{\varepsilon}{d - \varepsilon} |x - y| \quad \text{for } j = 0, 1, 2; \tag{2.11}$$

$$d\mathbb{Q}_\varepsilon \stackrel{\text{def}}{=} \phi_\varepsilon(0, \omega) d\mathbb{P} \text{ defines a probability measure equivalent to } \mathbb{P}. \tag{2.12}$$

Proof. The measurability property (2.7) is obvious. Due to assumption (2.4) we find that $\phi_\varepsilon(\cdot, \omega) \in C^2(\mathbb{R}^d)$ for all $0 \le \varepsilon \le 1, \omega \in \Omega$. The statements (2.9)–(2.11) follow by direct inspection of the formula (2.6) and property (2.12) is a direct consequence of (2.9) and the fact that $E^{\mathbb{P}}[\phi_\varepsilon(0, \omega)] = 1$ for all $0 \le \varepsilon \le 1$. Finally observe that for all $0 \le \varepsilon \le 1, \phi_\varepsilon \in \mathrm{Lip}_3^2(\mathbb{R}^d \times \Omega)$. □

Furthermore, we consider a measurable function $h(x, \omega) = (h_{i,j}(x, \omega))_{i,j=1,\dots,d}, x \in \mathbb{R}^d, \omega \in \Omega$, with values in the space of skew-symmetric $d \times d$-matrices, i.e.

$$h_{i,j}(x, \omega) = -h_{j,i}(x, \omega) \quad \text{for all } x \in \mathbb{R}^d, \omega \in \Omega \text{ and } i, j = 1, \dots, d, \tag{2.13}$$

and we assume that for all $i, j = 1, \dots, d$,

$$h_{i,j} \in \mathrm{Lip}_1^2(\mathbb{R}^d \times \Omega) \quad \text{and} \quad \mathcal{H}_A^{h_{i,j}} \subseteq \mathcal{H}_{A^{R/8}}^{\varphi} \tag{2.14}$$

for all Borel subsets A of $\mathbb{R}^d$. We then define the function $c = (c_1, \dots, c_d)^T : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}^d$ as

$$c_i(x, \omega) \stackrel{\text{def}}{=} \frac{1}{8d^2} \sum_{j=1}^{d} \partial_j h_{i,j}(x, \omega), \qquad \text{for all } x \in \mathbb{R}^d, \omega \in \Omega \text{ and } i = 1, \dots, d. \tag{2.15}$$

By direct inspection of the definition (2.15) and using the properties mentioned in (2.14) one easily finds

Lemma 2.2. *For all $x, y \in \mathbb{R}^d$ and $\omega \in \Omega$ we have that $c_i(\cdot, \omega) \in C^1(\mathbb{R}^d),\ i = 1, \dots, d$, and*

$$|c(x, \omega)| \le \frac{1}{8}, \qquad |c(x, \omega) - c(y, \omega)| \le \frac{1}{8}|x - y|, \qquad \nabla \cdot c(x, \omega) = 0. \tag{2.16}$$

Moreover, the following measurability property holds: for every Borel subset A of $\mathbb{R}^d$,

$$\mathcal{H}_A^c \subseteq \mathcal{H}_{A^{R/4}}^{\varphi}. \tag{2.17}$$

In addition to the above restrictions on the choices of c and φ we assume the following non-degeneracy condition to be satisfied:

$$E^{\mathbb{P}}[c(0, \omega)\varphi(0, \omega)] \ne 0. \tag{2.18}$$

The reason for this constraint will become clear when we construct the first example (see (3.4)). Note that since for all $i, j = 1, \dots, d, h_{i,j} \in \mathrm{Lip}_1^2(\mathbb{R}^d \times \Omega)$, both $h_{i,j}(x, \omega)$ and $\partial_j h_{i,j}(x, \omega)$ are bounded in absolute value by 1 for all $x \in \mathbb{R}^d, \omega \in \Omega$ and hence $E^{\mathbb{P}}[c_i(0, \omega)] \stackrel{(2.15)}{=} (8d^2)^{-1} \sum_{j=1}^d E^{\mathbb{P}}[\partial_j h_{i,j}(0, \omega)] = (8d^2)^{-1} \sum_{j=1}^d \partial_j E^{\mathbb{P}}[h_{i,j}(0, \omega)]$. Due to stationarity of the environment the latter expression equals zero. So, we find that

$$E^{\mathbb{P}}[c(0, \omega)] = 0. \tag{2.19}$$

This together with (2.18) implies that

$$\varphi(0, \omega) \not\equiv \text{const.} \quad \mathbb{P}\text{-a.s.} \tag{2.20}$$

For a possible example of functions φ and $h_{i,j}, i, j = 1, \dots, d$, with the properties mentioned in (2.4), (2.5) and (2.13), (2.14) respectively, such that (2.18) holds, we refer to Section 4.2. Now we are ready to introduce the family of drifts announced in the introduction of this section. In the notation of (2.6) and (2.15), for $0 \le \varepsilon \le 1$ and $\lambda \in [-1, 1]$, we define the drift

$$b_{\varepsilon,\lambda}(x, \omega) \stackrel{\text{def}}{=} \frac{\nabla \phi_\varepsilon(x, \omega)}{2\phi_\varepsilon(x, \omega)} + \varepsilon \left(\frac{c(x, \omega) + \lambda E^{\mathbb{P}}\left[\frac{c(0,\omega)}{\phi_\varepsilon(0,\omega)}\right]}{\phi_\varepsilon(x, \omega)} \right), \quad x \in \mathbb{R}^d, \omega \in \Omega. \tag{2.21}$$

Lemma 2.3. *There is an $0 < \varepsilon_0 \le 1$ such that for all $0 \le \varepsilon \le \varepsilon_0$ and $\lambda \in [-1, 1]$, the drift $b_{\varepsilon,\lambda}$ defined in (2.21) is finite range dependent with range R and for all $x, y \in \mathbb{R}^d$, $\omega \in \Omega$, the following holds:*

$$|b_{\varepsilon,\lambda}(x, \omega)| \le \varepsilon, \qquad |b_{\varepsilon,\lambda}(x, \omega) - b_{\varepsilon,\lambda}(y, \omega)| \le |x - y|. \tag{2.22}$$

The proof of Lemma 2.3 is given in Section 4.1.

For $0 \le \varepsilon \le \varepsilon_0$, $\lambda \in [-1, 1]$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$ we denote with $P_{x,\omega}^{\varepsilon,\lambda}$ the unique solution of the martingale problem starting in x at time 0 and attached to

$$\mathcal{L}^{\omega,\varepsilon,\lambda} = \frac{1}{2}\Delta + b_{\varepsilon,\lambda}(\cdot, \omega) \cdot \nabla, \tag{2.23}$$

see Corollary 5.4.29 in [8] for the existence of a unique solution of the above martingale problem. According to the definition (1.6) we denote by $P_0^{\varepsilon,\lambda}$ the *annealed* law $\mathbb{P} \times P_{0,\omega}^{\varepsilon,\lambda}$. We know that the process of the environment viewed from the particle associated to the diffusion $P_{0,\omega}^{\varepsilon,\lambda}$, which is defined in analogy to (A.3), has $\mathbb{Q}_\varepsilon$ (see (2.12)) as an invariant ergodic measure equivalent to $\mathbb{P}$ and

$$P_0^{\varepsilon,\lambda}\text{-a.s.}, \quad \frac{X_t}{t} \stackrel{t \to \infty}{\longrightarrow} v_{\varepsilon,\lambda} \stackrel{\text{def}}{=} E^{\mathbb{Q}_\varepsilon}\left[b_{\varepsilon,\lambda}(0, \omega)\right], \tag{2.24}$$

see Theorem A.1 in Appendix A. For the expected local drift under the static measure $\mathbb{P}$ we write

$$d_{\varepsilon,\lambda} \stackrel{\text{def}}{=} E^{\mathbb{P}}\left[b_{\varepsilon,\lambda}(0, \omega)\right]. \tag{2.25}$$

3. The examples

In this section we are going to provide examples of diffusions in random environment with arbitrarily small, non-vanishing expected local drift but without ballistic behavior. A slight modification of this family of diffusions gives us then examples of ballistic behavior, but where the expected local drift under the static measure vanishes or even has an opposite direction to the limiting velocity.

In order to shorten notation let us introduce the continuous function

$$g : [0, 1] \longrightarrow [1, 3]; \quad \varepsilon \mapsto g(\varepsilon) \stackrel{\text{def}}{=} E^{\mathbb{P}}\left[\phi_\varepsilon(0, \omega)^{-1}\right] \stackrel{\text{Jensen}}{\ge} E^{\mathbb{P}}\left[\phi_\varepsilon(0, \omega)\right]^{-1} \stackrel{(2.12)}{=} 1, \tag{3.1}$$

where $g \le 3$ follows from the uniform lower bound on ϕ_ε given in (2.9).

Now we are ready to provide our first examples. Recall the definition of ε_0 in Lemma 2.3.

Theorem 3.1 ($\lambda = 0$). *There exists an $0 < \varepsilon_1 \le \varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_1$ we have that $v_{\varepsilon,0} = 0$ but $d_{\varepsilon,0} \ne 0$.*

Proof. Let us begin with the calculation of the limiting velocity. Let $\lambda = 0$ and $0 \le \varepsilon \le \varepsilon_0$. Then

$$v_{\varepsilon,0} \overset{(2.24)}{=} E^{\mathbb{Q}_\varepsilon}\left[b_{\varepsilon,0}(0,\omega)\right] \overset{(2.12),(2.21)}{=} E^{\mathbb{P}}\left[\frac{1}{2}\nabla\phi_\varepsilon(0,\omega)\right] + \varepsilon E^{\mathbb{P}}\left[c(0,\omega)\right] = 0, \tag{3.2}$$

where in the last equality we used (2.19) and that $E^{\mathbb{P}}\left[\nabla\phi_\varepsilon(0,\omega)\right] = \nabla E^{\mathbb{P}}\left[\phi_\varepsilon(0,\omega)\right] = 0$, which can be shown by arguments similar to those in the derivation of (2.19). For the local drift under the static measure we have:

$$d_{\varepsilon,0} \overset{(2.25)}{=} E^{\mathbb{P}}\left[b_{\varepsilon,0}(0,\omega)\right] \overset{(2.21)}{=} \frac{1}{2}E^{\mathbb{P}}\left[\nabla \log\left(\phi_\varepsilon(0,\omega)\right)\right] + \varepsilon E^{\mathbb{P}}\left[\frac{c(0,\omega)}{\phi_\varepsilon(0,\omega)}\right].$$

Again by arguments similar to those leading to (2.19) we find that the first term on the right-hand side of the second equality above vanishes and hence

$$d_{\varepsilon,0} = \varepsilon E^{\mathbb{P}}\left[\frac{c(0,\omega)}{\phi_\varepsilon(0,\omega)}\right] \overset{(2.6)}{=} \varepsilon\left(1 + \frac{\varepsilon}{d}E^{\mathbb{P}}\left[\varphi(0,\omega)\right]\right) E^{\mathbb{P}}\left[\frac{c(0,\omega)}{1+\frac{\varepsilon}{d}\varphi(0,\omega)}\right]. \tag{3.3}$$

Using the identity (3.3) one can check by straightforward computations that

$$d_{0,0} = 0, \qquad \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} d_{\varepsilon,0} = E^{\mathbb{P}}[c(0,\omega)] \overset{(2.19)}{=} 0$$

and

$$\begin{aligned}\left.\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=0} d_{\varepsilon,0} &= \frac{2}{d}E^{\mathbb{P}}\left[\varphi(0,\omega)\right]E^{\mathbb{P}}\left[c(0,\omega)\right] - \frac{2}{d}E^{\mathbb{P}}\left[\varphi(0,\omega)c(0,\omega)\right]\\ &\overset{(2.19)}{=} -\frac{2}{d}E^{\mathbb{P}}\left[\varphi(0,\omega)c(0,\omega)\right] \overset{(2.18)}{\neq} 0.\end{aligned} \tag{3.4}$$

Since $d_{\varepsilon,0}$ is continuous in $\varepsilon \in [0,1]$, it then follows from the above computations that there exists an $0 < \varepsilon_1 \le \varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_1$, $d_{\varepsilon,0} \neq 0$ but at the same time we have that $v_{\varepsilon,0} = 0$, as shown in (3.2). □

Recall the definition of ε_1 in Theorem 3.1.

Theorem 3.2. *For all $0 < \varepsilon < \varepsilon_1$ and $\lambda \in (-1/3, 0)$ there exists a constant $\gamma > 0$ depending on ε and λ such that $d_{\varepsilon,\lambda} = -\gamma v_{\varepsilon,\lambda} \neq 0$.*

Proof. Recall the definition of g given in (3.1). As a consequence of Theorem 3.1 we know that for all $0 < \varepsilon < \varepsilon_1$ and $\lambda \in [-1,1] \setminus \{0\}$, the limiting velocity is not equal to zero. Indeed,

$$v_{\varepsilon,\lambda} \overset{(2.24)}{=} E^{\mathbb{Q}_\varepsilon}\left[b_{\varepsilon,\lambda}(0,\omega)\right] \overset{(2.21),(3.3)}{=} v_{\varepsilon,0} + \lambda d_{\varepsilon,0} \overset{(3.2)}{=} \lambda d_{\varepsilon,0} \neq 0. \tag{3.5}$$

Furthermore, for all $\lambda \in (-1/3, 0)$ and $0 < \varepsilon \le \varepsilon_0$, see Lemma 2.3 for the definition of ε_0, the expected local drift under the static measure $\mathbb{P}$ equals

$$d_{\varepsilon,\lambda} \overset{(2.25),(2.21),(3.3)}{=} d_{\varepsilon,0}\left(1 + \lambda g(\varepsilon)\right) \overset{(3.5)}{=} \lambda^{-1} v_{\varepsilon,\lambda}\left(1 + \lambda g(\varepsilon)\right) = -\gamma v_{\varepsilon,\lambda} \tag{3.6}$$

with $\gamma \overset{\text{def}}{=} -\lambda^{-1}(1 + \lambda g(\varepsilon)) > 0$, where we used that $\lambda \in (-1/3, 0)$ and $g \le 3$. □

Remark 3.3. We know that in the one-dimensional discrete setting, where we consider random walks in an i.i.d. random environment, the limiting velocity cannot have an opposite direction to the expected local drift under the static measure, see Remark on page 211 in [23]. In the case of one-dimensional diffusions in random environment as described in the Introduction, i.e. diffusions generated by operators of the form (1.5), the same holds true. Indeed, by Proposition 2.7 and formula (2.77) in [6] we know that

$$E^{\mathbb{P}}[b(0,\omega)] > 0 \iff P_0\text{-a.s.}, \lim_{t\to\infty} X_t = +\infty,$$

which implies that

$$P_0\text{-a.s.}, \quad \liminf_{t\to\infty} \frac{X_t}{t} \geq 0. \quad \square$$

In the next theorem we will see that by examining more carefully the formula for the expected drift under the static measure $d_{\varepsilon,\lambda} = d_{\varepsilon,0}(1+\lambda g(\varepsilon))$, which was derived in (3.6), we can find examples of diffusions with vanishing expected local drift but still with ballistic behavior. Recall the definition of ε_1 in Theorem 3.1.

Theorem 3.4. *For all $0 < \varepsilon < \varepsilon_1$ we find a $\lambda \in [-1, -1/3]$ depending on ε such that $v_{\varepsilon,\lambda} \neq 0$ and $d_{\varepsilon,\lambda} = 0$.*

Proof. Recall the definition of g given in (3.1). Let $0 < \varepsilon < \varepsilon_1$ and choose $\lambda \overset{\text{def}}{=} -g(\varepsilon)^{-1} \in [-1, -1/3]$. Then $v_{\varepsilon,\lambda} \neq 0$, which is shown in (3.5), and $d_{\varepsilon,\lambda} = d_{\varepsilon,0}(1+\lambda g(\varepsilon)) = 0$. $\square$

4. Auxiliary results

In this section we give the proof of Lemma 2.3 and provide a concrete example of functions φ and $h_{i,j}$, $i, j = 1, \ldots d$, with the properties given in (2.4), (2.5) and (2.13), (2.14) respectively, such that (2.18) holds. These functions are involved in the construction of the family of drifts defined in (2.21).

4.1. *Proof of* Lemma 2.3

It follows from the definition (2.21) and the measurability property (2.7) and (2.17) that for all $0 \leq \varepsilon \leq 1$ and $\lambda \in [-1, 1]$, $\mathcal{H}_A^{b_{\varepsilon,\lambda}} \subseteq \mathcal{H}_{A^{R/4}}^{\varphi}$, and hence the drift $b_{\varepsilon,\lambda}$ satisfies the finite range dependence condition with range R due to the finite range dependence of φ with range $R/2$ mentioned in (2.5). With the help of Lemmas 2.1 and 2.2 we find that for all $0 \leq \varepsilon \leq 1$, $\lambda \in [-1, 1]$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\begin{aligned} |b_{\varepsilon,\lambda}(x,\omega)| &\overset{(2.9)}{\leq} \frac{d+\varepsilon}{d-\varepsilon}\left\{\frac{1}{2}|\nabla\phi_\varepsilon(x,\omega)| + \varepsilon|c(x,\omega)| + \frac{d+\varepsilon}{d-\varepsilon}\varepsilon|\lambda| E^{\mathbb{P}}[|c(0,\omega)|]\right\} \\ &\overset{(2.10),(2.16)}{\leq} \varepsilon\frac{d+\varepsilon}{d-\varepsilon}\left\{\frac{1}{2}\frac{d}{d-\varepsilon} + \frac{1}{8} + \frac{1}{8}\frac{d+\varepsilon}{d-\varepsilon}\right\} \leq \varepsilon, \end{aligned}$$

where the last inequality holds for all $0 \leq \varepsilon \leq \delta_1$ with $\delta_1 > 0$ small enough. It remains to prove the Lipschitz property in (2.22). For the remainder of the proof let us define for all $0 \leq \varepsilon \leq 1$ and $\lambda \in [-1, 1]$,

$$\alpha_{\varepsilon,\lambda} \overset{\text{def}}{=} \lambda E^{\mathbb{P}}\left[\frac{c(0,\omega)}{\phi_\varepsilon(0,\omega)}\right] \overset{(2.9),(2.16)}{\leq} \vartheta_0, \tag{4.1}$$

for some constant $\vartheta_0 > 0$. For ε, λ as above, x, y in $\mathbb{R}^d$ and ω in Ω, we obtain

$$\begin{aligned}
&|b_{\varepsilon,\lambda}(x,\omega) - b_{\varepsilon,\lambda}(y,\omega)| \\
&\leq \frac{1}{2\phi_\varepsilon(x,\omega)\phi_\varepsilon(y,\omega)} \left\{ |\nabla\phi_\varepsilon(x,\omega)\phi_\varepsilon(y,\omega) - \nabla\phi_\varepsilon(y,\omega)\phi_\varepsilon(x,\omega)| \right. \\
&\quad \left. + 2\varepsilon\, |c(x,\omega)\phi_\varepsilon(y,\omega) - c(y,\omega)\phi_\varepsilon(x,\omega)| + 2\varepsilon \left|\alpha_{\varepsilon,\lambda}\right| |\phi_\varepsilon(y,\omega) - \phi_\varepsilon(x,\omega)| \right\} \\
&\stackrel{\text{def}}{=} \frac{f_\varepsilon(x,y,\omega) + 2\varepsilon g_\varepsilon(x,y,\omega) + 2\varepsilon|\alpha_{\varepsilon,\lambda}| h_\varepsilon(x,y,\omega)}{2\phi_\varepsilon(x,\omega)\phi_\varepsilon(y,\omega)} \\
&\stackrel{(2.9)}{\leq} \frac{9}{2}\left(f_\varepsilon(x,y,\omega) + 2\varepsilon g_\varepsilon(x,y,\omega) + 2\varepsilon|\alpha_{\varepsilon,\lambda}| h_\varepsilon(x,y,\omega) \right). \qquad (4.2)
\end{aligned}$$

We also find that $f_\varepsilon(x, y, \omega)$ is smaller than or equal to

$$\begin{aligned}
&|\nabla\phi_\varepsilon(x,\omega)|\, |\phi_\varepsilon(y,\omega) - \phi_\varepsilon(x,\omega)| + |\phi_\varepsilon(x,\omega)|\, |\nabla\phi_\varepsilon(x,\omega) - \nabla\phi_\varepsilon(y,\omega)| \\
&\quad \stackrel{(2.9)–(2.11)}{\leq} d\left(\frac{\varepsilon}{d-\varepsilon}\right)^2 |x-y| + \frac{d+\varepsilon}{d-\varepsilon} \cdot \frac{d\varepsilon}{d-\varepsilon}|x-y| \leq \varepsilon\vartheta_1|x-y|, \qquad (4.3)
\end{aligned}$$

for a positive constant ϑ_1, which depends on the upper bounds given in (2.9) and (2.10). By an analogous computation using (2.9)–(2.11) and (2.16) one can show that for some constant $\vartheta_2 > 0$,

$$g_\varepsilon(x, y, \omega) \leq \vartheta_2|x - y|. \qquad (4.4)$$

Finally, from (2.10) and (2.11) follows that there is a constant $\vartheta_3 > 0$ such that

$$h_\varepsilon(x, y, \omega) \leq \vartheta_3|x - y|. \qquad (4.5)$$

Collecting (4.1)–(4.5), we find the Lipschitz property (2.22) for all $0 \leq \varepsilon \leq \delta_2$ with $\delta_2 > 0$ small enough. Our claim then follows with ε_0 equal to the minimum of δ_1 and δ_2. This finishes the proof of Lemma 2.3.

4.2. *A concrete example*

A possible example of functions φ and $h_{i,j}$, $i, j = 1, \ldots, d$, with the properties mentioned in (2.4), (2.5) and (2.13), (2.14) respectively, such that (2.18) holds, can be constructed as follows. As a random environment $(\Omega, \mathcal{A}, \mathbb{P})$ we consider a canonical Poisson point process on $\mathbb{R}^d$ with constant intensity and for all $x \in \mathbb{R}^d$, $\omega \in \Omega$ and a Borel subset $A \subseteq \mathbb{R}^d$ we define the group of transformation on Ω as $\tau_x(\omega)(A) \stackrel{\text{def}}{=} \omega(x + A)$, where $x + A \stackrel{\text{def}}{=} \{x + a \mid a \in A\}$. Assume for the moment that we have a function $\varphi \in \mathrm{Lip}_1^3(\mathbb{R}^d \times \Omega)$ with $\partial_i\varphi \in \mathrm{Lip}_1^2(\mathbb{R}^d \times \Omega)$ satisfying the finite range dependence condition with range $R/2$ and (2.20). A possible skew-symmetric matrix $(h_{i,j}(x, \omega))_{i,j=1,\ldots,d}$, $x \in \mathbb{R}^d$, $\omega \in \Omega$ is defined as follows:

$h_{i,i}(x, \omega) = 0$, for $i = 1, \ldots, d$;

$h_{i,1}(x, \omega) = \partial_i\varphi(x, \omega)$ and $h_{1,i}(x, \omega) = -\partial_i\varphi(x, \omega)$, for $i = 2, \ldots, d$;

$(h_{i,j}(x, \omega))$, $i, j = 2, \ldots, d$, is a $(d - 1) \times (d - 1)$-dimensional skew-symmetric matrix with entries which fulfill (2.14).

One then easily sees that the entries $h = (h_{i,j})$, $i, j = 1, \ldots, d$, satisfy (2.14) and hence the function c defined via the formula (2.15) fulfills the properties in Lemma 2.2. The following lines show that condition (2.18) holds as well. Indeed assume the contrary, then:

$$
\begin{aligned}
0 &= E^{\mathbb{P}}\left[c_1(0,\omega)\varphi(0,\omega)\right] = -\frac{1}{8d^2}\sum_{j=2}^{d} E^{\mathbb{P}}\left[\left(\partial_j^2\varphi(0,\omega)\right)\varphi(0,\omega)\right] \\
&= -\frac{1}{8d^2}\sum_{j=2}^{d} E^{\mathbb{P}}\left[\partial_j\left(\partial_j\varphi(0,\omega)\varphi(0,\omega)\right)\right] + \frac{1}{8d^2}\sum_{j=2}^{d} E^{\mathbb{P}}\left[\left(\partial_j\varphi(0,\omega)\right)^2\right] \\
&= -\frac{1}{8d^2}\sum_{j=2}^{d} \partial_j E^{\mathbb{P}}\left[\partial_j\varphi(0,\omega)\varphi(0,\omega)\right] + \frac{1}{8d^2}\sum_{j=2}^{d} E^{\mathbb{P}}\left[\left(\partial_j\varphi(0,\omega)\right)^2\right] \\
&= \frac{1}{8d^2}\sum_{j=2}^{d} E^{\mathbb{P}}\left[\left(\partial_j\varphi(0,\omega)\right)^2\right],
\end{aligned}
$$

where we used stationarity in the last equality. This implies that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$
\partial_j\varphi(0,\omega) = 0, \quad \text{for all } j = 2,\dots,d, \tag{4.6}
$$

and hence by stationarity and continuity of $\partial_j\varphi(\cdot,\omega)$ for each $\omega \in \Omega$, we find that $\mathbb{P}$-a.s.,

$$
\partial_j\varphi(x,\omega) = 0, \quad \text{for all } x \in \mathbb{R}^d \text{ and } j = 2,\dots,d. \tag{4.7}
$$

It follows that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\varphi(x,\omega)$ is in fact a function of x_1 and ω only. In the notation $\bar{R}_2 = (0, R, 0, \dots, 0) \in \mathbb{R}^d$ we thus have that $\mathbb{P}$-a.s., $\varphi(0,\omega) = \varphi(\bar{R}_2,\omega)$ and since φ satisfies the finite range dependence condition with range $R/2$ we find that for all integers $n \geq 0$,

$$
E^{\mathbb{P}}\left[\varphi(0,\omega)^n\right] = E^{\mathbb{P}}\left[\prod_{k=0}^{n-1}\varphi(k\bar{R}_2,\omega)\right] \overset{\text{indep.}}{=} \prod_{k=0}^{n-1} E^{\mathbb{P}}\left[\varphi(k\bar{R}_2,\omega)\right] = E^{\mathbb{P}}\left[\varphi(0,\omega)\right]^n
$$

which is a contradiction to (2.20) and (2.18) must hold true.

We now come to the construction of a possible φ satisfying the above required conditions. Pick a measurable real-valued non-negative function $\zeta(x), x \in \mathbb{R}^d$, which is supported in a ball of radius $R/8$ and strictly positive on a set of positive Lebesgue measure. Then convolve the Poisson point process with ζ and truncate, let us say at 1, the new function, i.e. for $x \in \mathbb{R}^d$, $\omega \in \Omega$ we define

$$
\hat{\varphi}(x,\omega) \overset{\text{def}}{=} \left(\int_{\mathbb{R}^d}\zeta(x-y)\omega(\mathrm{d}y)\right) \wedge 1. \tag{4.8}
$$

After smoothing with a non-negative mollifier ρ belonging to $C^3(\mathbb{R}^d)$ which is also supported in a ball of radius $R/8$ we obtain a function

$$
\tilde{\varphi}(x,\omega) \overset{\text{def}}{=} \int_{\mathbb{R}^d}\hat{\varphi}(x-y,\omega)\rho(y)\mathrm{d}y, \quad x \in \mathbb{R}^d, \omega \in \Omega, \tag{4.9}
$$

that satisfies $\tilde{\varphi} \in \mathrm{Lip}_m^3(\mathbb{R}^d \times \Omega)$ with $\partial_i\tilde{\varphi} \in \mathrm{Lip}_m^2(\mathbb{R}^d \times \Omega)$, $i = 1,\dots d$, where $m > 0$ depends on the mollifier. A possible candidate for φ is then $\tilde{\varphi}/m$.

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Appendix A

A.1. Invariant ergodic measure

The aim of Appendix A is to show the existence of an invariant ergodic measure for the process of the environment viewed from the particle defined in (A.3) for a special class of diffusions in random environment and derive a formula for the limiting velocity of the diffusion, see Theorem A.1. We consider a class of diffusions that are Brownian motions which are perturbed by environment-dependent drifts of the form given in (A.1). Note that we will not assume any finite range dependence condition for the environment in order to derive the following results.

Let us consider a diffusion in random environment $\omega \in \Omega$ which is generated by the operator $\mathcal{L}^\omega$ given in (1.5) with a drift of the form

$$b(x,\omega) \stackrel{\text{def}}{=} \frac{\nabla\phi(x,\omega)}{2\phi(x,\omega)} + \frac{\Gamma(x,\omega)}{\phi(x,\omega)}, \quad x \in \mathbb{R}^d, \omega \in \Omega, \tag{A.1}$$

where $\phi \in \mathrm{Lip}^2_m(\mathbb{R}^d \times \Omega)$ for some $m > 0$, see below (2.2) for the definition of $\mathrm{Lip}^2_m(\mathbb{R}^d \times \Omega)$, such that

$$m^{-1} \le \inf_{x\in\mathbb{R}^d,\omega\in\Omega} \phi(x,\omega) \le \sup_{x\in\mathbb{R}^d,\omega\in\Omega} \phi(x,\omega) \le m. \tag{A.2}$$

Moreover, we assume that $\phi(0,\omega)$ is a probability density with respect to the static measure of the environment $\mathbb{P}$ and $\Gamma = (\Gamma_1,\ldots,\Gamma_d)^T : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R}^d$ is a measurable function such that $\Gamma_i \in \mathrm{Lip}^0_l(\mathbb{R}^d \times \Omega)$ for some $l \ge 0$ and $\Gamma_i(\cdot,\omega) \in C^1(\mathbb{R}^d)$, $i = 1,\ldots,d$, with $\nabla \cdot \Gamma(x,\omega) = 0$ for all $x \in \mathbb{R}^d, \omega \in \Omega$. Note that similarly as in the proof of (2.22) one can show that the drift b given in (A.1) satisfies the Lipschitz condition (1.3) and its Euclidean norm is uniformly bounded. Thus the martingale problem attached to (1.5) with the above drift starting from $x \in \mathbb{R}^d$ at time 0 is well posed. Let in what follows $P_{x,\omega}$, $x \in \mathbb{R}^d, \omega \in \Omega$, denote its unique solution and recall that $(X_t)_{t\ge0}$ stands for the coordinate process on $C(\mathbb{R}_+,\mathbb{R}^d)$. We can associate the canonical process on Ω defined by the environment $\omega \in \Omega$ as seen by an observer sitting on the particle, i.e.

$$\begin{cases} \omega(t) = \tau_{X_t}(\omega), \\ \omega(0) = \omega \in \Omega. \end{cases} \tag{A.3}$$

This map induces a measure Q_ω on the space of trajectories in Ω starting from $\omega \in \Omega$ which can be shown to be a Markov process on Ω with semigroup given by

$$\mathcal{Q}_t \tilde{f}(\omega) \stackrel{\text{def}}{=} E_{0,\omega}\left[\tilde{f}(\tau_{X_t}(\omega))\right], \quad t \ge 0,$$

for all bounded measurable functions $\tilde{f}$ on Ω. Now we state the main theorem of Appendix A.

Theorem A.1. *The probability measure $d\mathbb{Q} \stackrel{\text{def}}{=} \phi(0,\omega)d\mathbb{P}$ is an invariant ergodic measure for the Markov family $\{Q_\omega\}_{\omega\in\Omega}$ which is equivalent to the static measure $\mathbb{P}$ and*

$$\mathbb{P} \times P_{0,\omega} = P_0\text{-}a.s., \quad \lim_{t\to\infty} \frac{X_t}{t} = v \stackrel{\text{def}}{=} E^{\mathbb{Q}}[b(0,\omega)]. \tag{A.4}$$

Proof. Since for all $x \in \mathbb{R}^d, \omega \in \Omega$, $\frac{1}{2}\Delta\phi(x,\omega) = \nabla \cdot (b(x,\omega)\phi(x,\omega))$ holds true, recall that $\nabla \cdot \Gamma(x,\omega) = 0$, and $\phi(0,\omega)$ is a probability density with respect to $\mathbb{P}$, we know that $\mathbb{Q}$ is an invariant probability measure for the family $\{Q_\omega\}_{\omega\in\Omega}$, see for instance Section 6 in [27], Section 6 in [11]. Due to the uniform ellipticity of the diffusion matrix, which is in fact the identity matrix (see (1.5)), it can be shown that $\mathbb{Q}$ is also ergodic. Indeed, a necessary and sufficient condition for ergodicity is that whenever a function $\tilde{g}$ on Ω which is square integrable with respect to the measure $\mathbb{Q}$ satisfies

$$\mathcal{Q}_t \tilde{g} = \tilde{g}, \tag{A.5}$$

$\mathbb{Q}$-a.s. for all $t > 0$, then $\tilde{g}$ has to be constant $\mathbb{Q}$-a.s., see Theorem 3.2.4 in [2]. To show this we multiply both sides of (A.5) by $\tilde{g}$ and average over the environment with respect to the measure $\mathbb{Q}$. After some manipulations using the invariance property of $\mathbb{Q}$ we find that (A.5) leads to

$$\begin{aligned} &E^{\mathbb{Q}}\left[E_{0,\omega}\left[\left(\tilde{g}(\tau_{X_t}(\omega)) - \tilde{g}(\omega)\right)^2\right]\right] \\ &\quad = E^{\mathbb{P}}\left[\phi(0,\omega)\int_{\mathbb{R}^d} p_\omega(t,0,y)\left(\tilde{g}(\tau_y(\omega)) - \tilde{g}(\omega)\right)^2 \mathrm{d}y\right] = 0. \end{aligned} \tag{A.6}$$

Due to the fact that $p_\omega(t,0,y) > 0$, for all $\omega \in \Omega$, $y \in \mathbb{R}^d$, $t > 0$, see Theorem 1 on page 67 in [7], and $\mathbb{P}[\phi(0,\omega) \geq m^{-1}] = 1$, which comes from the uniform lower bound given in (A.2), we can deduce from (A.6) that

$$\mathbb{P}\text{-a.s.,} \quad \tilde{g}(\tau_y(\omega)) = \tilde{g}(\omega) \quad \text{for a.e. } y \in \mathbb{R}^d. \tag{A.7}$$

An application of the spatial ergodic theorem shown in [4], see Theorem 10 on page 694, and the stationarity of the environment then show that (A.7) holds true for all $y \in \mathbb{R}^d$. From the assumed ergodicity of the family of transformations $\{\tau_x : x \in \mathbb{R}^d\}$, mentioned above (1.1), it follows that $\tilde{g}$ is constant $\mathbb{P}$-a.s. and since $\mathbb{Q}$ is a measure equivalent to $\mathbb{P}$ due to (A.2), $\tilde{g}$ is also constant $\mathbb{Q}$-almost surely. This shows the ergodicity of $\mathbb{Q}$. Therefore, for each $\omega \in \Omega$ we have that for all bounded measurable functions $\tilde{f}$ on Ω,

$$\mathbb{Q} \times Q_\omega\text{-a.s.,} \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \tilde{f}(\omega(s))\mathrm{d}s = E^{\mathbb{Q}}[\tilde{f}], \tag{A.8}$$

which follows from Theorem 3.3.1 in [2]. By definition of Q_ω we have that under the measure Q_ω the process $(\tilde{f}(\omega(s)))_{s\geq 0}$ has the same law as the process $(f(X_s,\omega))_{s\geq 0}$ under the measure $P_{0,\omega}$, where we define $f(x,\omega) \stackrel{\text{def}}{=} \tilde{f}(\tau_x(\omega))$ for all $x \in \mathbb{R}^d$, $\omega \in \Omega$, according to the definition (1.1). Thus, (A.8) is equivalent to

$$\mathbb{Q} \times P_{0,\omega}\text{-a.s.,} \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_s,\omega)\mathrm{d}s = E^{\mathbb{Q}}\left[f(0,\omega)\right]. \tag{A.9}$$

Since for $\omega \in \Omega$, $P_{0,\omega}$-a.s., $X_t = \int_0^t b(X_s,\omega)\mathrm{d}s + W_t$, for all $t \geq 0$, for some Brownian motion $(W_t)_{t\geq 0}$, (A.9) together with the law of large numbers for Brownian motion, see page 104 in [8], yield

$$\mathbb{Q} \times P_{0,\omega}\text{-a.s.,} \quad \lim_{t\to\infty} \frac{X_t}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t b(X_s,\omega)\mathrm{d}s = E^{\mathbb{Q}}\left[b(0,\omega)\right]. \tag{A.10}$$

Since $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, (A.10) holds true $\mathbb{P} \times P_{0,\omega}$-almost surely. This concludes the proof of Theorem A.1. □

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